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Journal of Approximation Theory 127 (2004) 207–222
Journal of Approximation Theory
http://www.elsevier.com/locate/jat

On constants in some inequalities for intermediate derivatives on a finite interval

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Received 17 July 2003; accepted in revised form 2 March 2004

Communicated by Borislav Bojanov

Abstract

Let L_q $(1 \leqslant q < \infty)$ be the space of functions f measurable on $I = [-1, 1]$ and integrable to the power q, with norm

$$||f||_q = \left\{ \int_{-1}^{1} |f(x)|^q \, dx \right\}^{\frac{1}{q}}.$$

L_∞ is the space of functions measurable on I with norm

$$||f||_\infty = \operatorname*{esssup}_{|x| \leqslant 1} |f(x)| < \infty.$$

We denote by AC the set of all functions absolutely continuous on I. For $n \in N$, $q \in [1, \infty]$ we set

$$W_{n,q} = \{f : f^{(n-1)} \in AC, \ f^{(n)} \in L_q\}.$$

In this paper, we consider the problem of accuracy of constants A, B in the inequalities

$$||f^{(m)}||_q \leqslant A||f||_p + B||f^{(m+k+1)}||_r, \quad m \in N, \quad k \in W; \qquad p, q, r \in [1, \infty], \ f \in W_{m+k+1,r}. \tag{1}$$

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.03.003

1. Introduction

As it follows from our subsequent considerations $\forall p,q,r\in[1,\infty]$ there are constants $A,B>0$, such that estimate (1) holds $\forall f\in W_{m+k+1,r}$. We set

$$A_{m+k+1,m}(p,q,r)=\inf_A A,$$

where inf is taken over all $A>0$ such that for some $B>0$ inequality (1) holds $\forall f\in W_{m+k+1,r}$. We set also

$$B_{m+k+1,m}(p,q,r)=\inf B,$$

$$B{:}A=A_{m+k+1,m}(p,q,r),$$

where inf is taken over all constants $B>0$ such that (1) holds $\forall f\in W_{m+k+1,r}$ with $A=A_{m+k+1,m}(p,q,r)$. It was proved in [3] that $\forall m\in N$, $1\leqslant p,q,r\leqslant\infty$ the following relations are valid:

$$2^{m+\frac{1}{q}-\frac{1}{p}-1}m!=A_{m+1,m}(\infty,q,r)2^{-\frac{1}{p}}\leqslant A_{m+1,m}(p,q,r)$$
$$\leqslant A_{m+1,m}(1,q,r)2^{1-\frac{1}{p}}=2^m m!2^{\frac{1}{q}-\frac{1}{p}}. \tag{2}$$

In this paper, we find explicit expression for $A_{m+k+1,m}(2,\infty,r)$ $(m\in N,\ k\in W,\ 1\leqslant r\leqslant\infty)$. In some cases we find estimates of $A_{m+k+1,m}(p,q,r)$ from above.

2. The main tools

Our main tool is one integral representation of functions related to orthogonal polynomials.

Let p be a function defined on I such that (1) $p\in AC$, (2) $p(x)\geqslant 0$ on I, (3) $p(x)>0$ on a set of positive Lebesgue measure. Such a function will be called a weight function. Let H_n $(n\in W)$ be the set of all algebraic polynomials of degree at most n. We set

$$L_{p(\cdot)}=\left\{f{:}f \text{ is measurable on } I,\int_{-1}^{1}p(x)|f(x)|\,dx<\infty\right\}.$$

Let $\{\omega_{k;p(\cdot)}\}_0^\infty$ $(\omega_{k;p(\cdot)}\in H_k,\ k\in W)$ designate the system of algebraic polynomials orthonormal on I with respect to the weight function p. For $f\in L_{p(\cdot)}$ we set

$$c_{k;p(\cdot)}(f)=\int_{-1}^{1}p(u)f(u)\omega_{k;p(\cdot)}(u)\,du,\quad k\in W,$$

$$S_{m;p(\cdot)}(f)=\sum_{k=0}^{m}c_{k;p(\cdot)}(f)\omega_{k;p(\cdot)},\quad m\in W.$$

We note that

$$S_{m;p(\cdot)}(f)=f\quad\text{if } f\in H_m\quad(m\in W). \tag{3}$$

For $r \in N$, $x, u \in I$ we set

$$L_{r;p(\cdot)}(u,x) = p(u) \sum_{k=0}^{r-1} \omega_{k;p(\cdot)}(u)\omega_{k;p(\cdot)}(x).$$

For $x, t \in I$, $r \in N$ we introduce also

$$N_{r;p(\cdot)}(x,t) = \begin{cases} \int_t^1 (t-u)^{r-1} L_{r;p(\cdot)}(u,x)\,du, & -1 \leqslant x \leqslant t \leqslant 1, \\ -\int_{-1}^t (t-u)^{r-1} L_{r;p(\cdot)}(u,x)\,du, & -1 \leqslant t < x \leqslant 1. \end{cases} \tag{4}$$

Lemma 1. *If* $r \in N$, $x_0 \in I$, $f^{(r-1)} \in AC$, *then*

$$f(x_0) - S_{r-1;p(\cdot)}(f;x_0) = \frac{(-1)^r}{\Gamma(r)} \int_{-1}^1 f^{(r)}(t) N_{r;p(\cdot)}(x_0,t)\,dt. \tag{5}$$

Proof. Since the functional $\phi_{x_0}(f) = f(x_0) - S_{r-1;p(\cdot)}(f;x_0)$ according to (3) annihilates all elements of H_{r-1}, hence due to the Peano's theorem (see [5, p. 70]) we have

$$\phi_{x_0}(f) = \int_{-1}^1 f^{(r)}(t)K(t)\,dt, \tag{6}$$

where

$$K(t) = \frac{1}{\Gamma(r)} \phi_{x_0;x}[(x-t)_+^{r-1}], \tag{7}$$

$$(x-t)_+^{r-1} = \begin{cases} (x-t)^{r-1}, & x \geqslant t, \\ 0, & x < t, \end{cases}$$

the notation $\phi_{x_0;x}[(x-t)_+^{r-1}]$ means that the functional ϕ_{x_0} is applied to $(x-t)_+^{r-1}$ considered as a function of x. Obviously,

$$\begin{aligned} \phi_{x_0;x}[(x-t)_+^{r-1}] &= (x_0-t)_+^{r-1} - \int_{-1}^1 L_{r;p(\cdot)}(u,x_0)(u-t)_+^{r-1}\,du \\ &= \begin{cases} (x_0-t)^{r-1} - \int_t^1 (u-t)^{r-1} L_{r;p(\cdot)}(u,x_0)\,du, & t < x_0, \\ -\int_t^1 (u-t)^{r-1} L_{r;p(\cdot)}(u,x_0)\,du, & t \geqslant x_0. \end{cases} \end{aligned} \tag{8}$$

It follows from (6)–(8) that

$$\begin{aligned} \phi_{x_0}(f) = &\frac{1}{\Gamma(r)} \Bigg\{ \int_{-1}^{x_0} f^{(r)}(t)(x_0-t)^{r-1}\,dt \\ &- \int_{-1}^1 f^{(r)}(t) \left(\int_t^1 (u-t)^{r-1} L_{r;p(\cdot)}(u,x_0)\,du \right) dt \Bigg\}. \end{aligned} \tag{9}$$

On the other hand, taking (4) and (3) into consideration we get

$$\begin{aligned}
&\frac{(-1)^r}{\Gamma(r)}\int_{-1}^{1} f^{(r)}(t)N_{r;p(\cdot)}(x_0,t)\,dt\\
&\quad=\frac{(-1)^r}{\Gamma(r)}\Big\{-\int_{-1}^{x_0} f^{(r)}(t)\left(\int_{-1}^{t}(t-u)^{r-1}L_{r;p(\cdot)}(u,x_0)\,du\right)dt\\
&\qquad+\int_{x_0}^{1} f^{(r)}(t)\left(\int_{t}^{1}(t-u)^{r-1}L_{r;p(\cdot)}(u,x_0)\,du\right)dt\\
&\quad=\frac{1}{\Gamma(r)}\int_{-1}^{x_0} f^{(r)}(t)\left(\int_{-1}^{1}(u-t)^{r-1}L_{r;p(\cdot)}(u,x_0)\,du\right)dt\\
&\qquad-\frac{1}{\Gamma(r)}\int_{-1}^{x_0} f^{(r)}(t)\left(\int_{t}^{1}(u-t)^{r-1}L_{r;p(\cdot)}(u,x_0)\,du\right)dt\\
&\qquad+\frac{(-1)^r}{\Gamma(r)}\int_{x_0}^{1} f^{(r)}(t)\left(\int_{t}^{1}(t-u)^{r-1}L_{r;p(\cdot)}(u,x_0)\,du\right)dt\\
&\quad=\frac{1}{\Gamma(r)}\int_{-1}^{x_0} f^{(r)}(t)(x_0-t)^{r-1}\,dt\\
&\qquad-\frac{1}{\Gamma(r)}\int_{-1}^{1} f^{(r)}(t)\left(\int_{t}^{1}(u-t)^{r-1}L_{r;p(\cdot)}(u,x_0)\,du\right)dt. \qquad (10)
\end{aligned}$$

Equalities (9) and (10) yield (5). Lemma 1 is proved. □

Lemma 2. *If $p(u)=p_m(u)=(1-u^2)^m$ $(m\in N)$, $k\in W$, $f^{(m-1)}\in AC$, then*

$$S_{k;p_m(\cdot)}(f^{(m)};x)=\int_{-1}^{1} f(u)\sum_{i=0}^{k} J_i^{(m,m)}(x)\,\frac{\sqrt{\Gamma(i+2m+1)}}{\sqrt{\Gamma(i+1)}}\,X_{i+m}(u)\,du. \qquad (11)$$

Here $\{X_k\}_0^\infty$ is the orthonormal on I system of Legendre polynomials, $\{J_k^{(m,m)}\}_0^\infty$ is the orthonormal on I with respect to the weight function $(1-u^2)^m$ system of Jacobi polynomials.

Proof. Obviously,

$$S_{k;p_m(\cdot)}(f^{(m)};x)=\int_{-1}^{1} f^{(m)}(u)L_{k+1;p_m(\cdot)}(u,x)\,du,$$

$$L_{k+1;p_m(\cdot)}(u,x)=(1-u^2)^m\sum_{i=0}^{k} J_i^{(m,m)}(u)J_i^{(m,m)}(x).$$

By integrating m times by parts we get

$$\begin{aligned}
S_{k;p_m(\cdot)}(f^{(m)};x)&=(-1)^m\int_{-1}^{1} f(u)\,\frac{d^m}{du^m}\\
&\quad\times\left((1-u^2)^m\sum_{i=0}^{k} J_i^{(m,m)}(u)J_i^{(m,m)}(x)\right)du. \qquad (12)
\end{aligned}$$

We will make use of the formula (see [9])

$$\frac{d^m}{du^m}((1-u^2)^m P_i^{(m,m)}(u)) = (-1)^m 2^m (m+i)(m+i-1)\cdots(i+1)P_{m+i}(u), \quad (13)$$

where $P_k(u) = \sqrt{\frac{2}{2k+1}}X_k(u)$ $(k\in W)$, $\{P_k^{(m,m)}\}_0^\infty$ is the system of Jacobi polynomials orthogonal on I with the weight $(1-u^2)^m$ and standardized by the condition $P_k^{(m,m)}(1) = \frac{\Gamma(k+m+1)}{k!\Gamma(m+1)}$. Equality (11) follows directly from (12) and (13). Lemma 2 is proved. □

3. Theorem 1

Theorem 1. *The following relation holds true if* $m\in N$, $k\in W$, $r\in[1,\infty]$:

$$A_{m+k+1,m}(2,\infty,r) = \frac{1}{2^{m+\frac{1}{2}}\Gamma(m+1)} \times \left(\sum_{i=0}^{k} \frac{\Gamma^2(i+2m+1)(2i+2m+1)}{(i!)^2}\right)^{\frac{1}{2}}. \quad (14)$$

Proof. Equality (11) implies $(k\in W,\ x\in I,\ f^{(m-1)}\in AC)$

$$|S_{k;p_m(\cdot)}(f^{(m)};x)| \leqslant ||f||_2 \left(\sum_{i=0}^{k} \frac{\Gamma(i+2m+1)}{\Gamma(i+1)}(J_i^{(m,m)}(x))^2\right)^{\frac{1}{2}},$$

which in turn leads to

$$||S_{k;p_m(\cdot)}(f^{(m)};x)||_\infty \leqslant ||f||_2 \left(\sum_{i=0}^{k} \frac{\Gamma(i+2m+1)}{\Gamma(i+1)}(J_i^{(m,m)}(1))^2\right)^{\frac{1}{2}}$$

$$= ||f||_2 \left(\sum_{i=0}^{k} \frac{\Gamma^2(i+2m+1)(2i+2m+1)}{(i!)^2}\right)^{\frac{1}{2}} \times \frac{1}{2^{m+\frac{1}{2}}\Gamma(m+1)}. \quad (15)$$

It follows from the equality

$$f^{(m)}(x) - S_{k;p_m(\cdot)}(f^{(m)};x) = \frac{(-1)^{k+1}}{\Gamma(k+1)} \int_{-1}^{1} f^{(m+k+1)} N_{k+1;p_m(\cdot)}(x,t)\,dt, \quad (16)$$

inequality (15) and Hölder's inequality that $\forall r \in [1, \infty]$ we have

$$A_{m+k+1,m}(2, \infty, r) \leqslant \frac{1}{2^{m+\frac{1}{2}}\Gamma(m+1)} \times \left(\sum_{i=0}^{k} \frac{\Gamma^2(i+2m+1)(2i+2m+1)}{(i!)^2}\right)^{\frac{1}{2}}. \tag{17}$$

We will prove now that in fact there is equal sign in inequality (17). Assume that there is $\varepsilon > 0$ such that $\forall f$ and $f \in W_{m+k+1,r}$ we have

$$||f^{(m)}||_\infty \leqslant ||f||_2 \left[\frac{1}{2^{m+\frac{1}{2}}\Gamma(m+1)}\left(\sum_{i=0}^{k} \frac{\Gamma^2(i+2m+1)(2i+2m+1)}{(i!)^2}\right)^{\frac{1}{2}} - \varepsilon\right] + C(m,k,r)||f^{(m+k+1)}||_r, \tag{18}$$

where $C(m,k,r)$ is a positive constant depending on m, k, r. We set

$$f_0(u) = \sum_{i=0}^{k} J_i^{(m,m)}(1)\sqrt{\frac{\Gamma(i+2m+1)}{\Gamma(i+1)}}\, X_{i+m}(u) \in H_{k+m}. \tag{19}$$

It follows from (3), (16), (19) that $\forall x \in I$ we have

$$f_0^{(m)}(x) = \int_{-1}^{1} f_0(u) \sum_{i=0}^{k} J_i^{(m,m)}(x) \frac{\sqrt{\Gamma(i+2m+1)}}{\sqrt{\Gamma(i+1)}} X_{i+m}(u)\, du,$$

therefore

$$f_0^{(m)}(1) = \frac{1}{2^{2m+1}\,\Gamma^2(m+1)} \sum_{i=0}^{k} \frac{\Gamma^2(i+2m+1)(2m+2i+1)}{(i!)^2}. \tag{20}$$

By applying (18) to $f_0(u)$ we get

$$|f_0^{(m)}(1)| \leqslant ||f_0||_2 \times \left[\frac{1}{2^{m+\frac{1}{2}}\Gamma(m+1)}\left(\sum_{i=0}^{k} \frac{\Gamma^2(i+2m+1)(2i+2m+1)}{(i!)^2}\right)^{\frac{1}{2}} - \varepsilon\right]$$

$$= \left(\sum_{i=0}^{k} (J_i^{(m,m)}(1))^2 \frac{\Gamma(i+2m+1)}{\Gamma(i+1)}\right)^{\frac{1}{2}} \times \left[\frac{1}{2^{m+\frac{1}{2}}\Gamma(m+1)}\left(\sum_{i=0}^{k} \frac{\Gamma^2(i+2m+1)(2i+2m+1)}{(i!)^2}\right)^{\frac{1}{2}} - \varepsilon\right]$$

$$=\frac{1}{2^{m+\frac{1}{2}}\Gamma(m+1)}\left(\sum_{i=0}^{k}\frac{\Gamma^2(i+2m+1)(2i+2m+1)}{(i!)^2}\right)^{\frac{1}{2}}$$

$$\times\left[\frac{1}{2^{m+\frac{1}{2}}\Gamma(m+1)}\left(\sum_{i=0}^{k}\frac{\Gamma^2(i+2m+1)(2i+2m+1)}{(i!)^2}\right)^{\frac{1}{2}}-\varepsilon\right]. \qquad (21)$$

Relations (20) and (21) lead to contradiction. Theorem 1 is proved. □

Remark 1. By using the weight function $p_{m,\alpha}(u)=(1-u^2)^{m+\alpha}$ $(m\in N,\ \alpha>-1)$ instead of $p_m(u)$ and following the same line of reasoning as for Theorem 1, we can find the best constant $A_{m+k+1,m}(2,\alpha;\infty,r)$ in the estimate

$$||f^{(m)}||_\infty \leqslant A||f||_{2,\alpha}+B||f^{(m+k+1)}||_r, \qquad (22)$$

where

$$||f||_{2,\alpha}=\left\{\int_{-1}^{1}(1-x^2)^\alpha f^2(x)\,dx\right\}^{\frac{1}{2}},\quad A_{m+k+1,m}(2,\alpha;\infty,r)=\inf_A A,$$

where inf is taken over all $A>0$ such that for some $B>0$ inequality (22) holds $\forall f\in W_{m+k+1,r}$. The formula for $A_{m+k+1,m}(2,\alpha;\infty,r)$ is

$$A_{m+k+1,m}(2,\alpha;\infty,r)$$

$$=\frac{1}{2^{m+\alpha+\frac{1}{2}}\Gamma(m+\alpha+1)}$$

$$\times\left(\sum_{i=0}^{k}\frac{\Gamma^2(i+2m+2\alpha+1)\Gamma(m+i+1)(2m+2\alpha+2i+1)}{\Gamma(i+m+2\alpha+1)(i!)^2}\right)^{\frac{1}{2}}.$$

4. Some estimates for $A_{m+k+1,m}(2,q,r)$

Theorem 2. *For $m\in N$, $k\in W$, $1\leqslant q,r\leqslant\infty$ the following estimate holds true:*

$$A_{m+k+1,m}(2,q,r)\leqslant\frac{2^{\frac{1}{q}-m-\frac{1}{2}}}{\Gamma(m+1)}\left(\sum_{i=0}^{k}\frac{\Gamma^2(i+2m+1)(2m+2i+1)}{(i!)^2}\right)^{\frac{1}{2}}. \qquad (23)$$

Proof. It follows from (11) and (16) that $\forall x \in I$ and $\forall f$, such that $f^{(m+k)} \in AC$ we have

$$f^{(m)}(x) = \int_{-1}^{1} f(u) \sum_{i=0}^{k} J_i^{(m,m)}(x) \frac{\sqrt{\Gamma(i+2m+1)}}{\sqrt{\Gamma(i+1)}} X_{i+m}(u)\, du + \frac{(-1)^{k+1}}{k!} \int_{-1}^{1} f^{(m+k+1)}(t) N_{k+1;p(\cdot)}(x,t)\, dt. \tag{24}$$

By applying Minkowski inequality we obtain from (24) that

$$\|f^{(m)}\|_q \leqslant \left\{ \int_{-1}^{1} \left| \int_{-1}^{1} f(u) \sum_{i=0}^{k} J_i^{(m,m)}(x) \times \frac{\sqrt{\Gamma(i+2m+1)}}{\sqrt{\Gamma(i+1)}} X_{i+m}(u)\, du \right|^q dx \right\}^{\frac{1}{q}} + \frac{1}{k!} \left\{ \int_{-1}^{1} \left| \int_{-1}^{1} f^{(m+k+1)}(t) N_{k+1;p(\cdot)}(x,t)\, dt \right|^q dx \right\}^{\frac{1}{q}}.$$

It follows from Hölder's inequality that

$$\begin{aligned}
\|f^{(m)}\|_q \leqslant & \left\{ \int_{-1}^{1} \left[\|f\|_2 \left(\sum_{i=0}^{k} \frac{\Gamma(i+2m+1)}{\Gamma(i+1)} (J_i^{(m,m)}(x))^2 \right)^{\frac{1}{2}} \right]^q dx \right\}^{\frac{1}{q}} \\
& + \frac{1}{k!} \|f^{(m+k+1)}\|_r \times \left[\int_{-1}^{1} \left(\int_{-1}^{1} |N_{k+1;p(\cdot)}(x,t)|^{r'} dt \right)^{\frac{q}{r'}} dx \right]^{\frac{1}{q}} \\
\leqslant & \|f\|_2 \left(\sum_{i=0}^{k} \frac{\Gamma(i+2m+1)}{\Gamma(i+1)} (J_i^{(m,m)}(1))^2 \right)^{\frac{1}{2}} 2^{\frac{1}{q}} \\
& + \frac{1}{k!} \|f^{(m+k+1)}\|_r \left[\int_{-1}^{1} \left(\int_{-1}^{1} |N_{k+1;p(\cdot)}(x,t)|^{r'} dt \right)^{\frac{q}{r'}} dx \right]^{\frac{1}{q}} \\
= & \|f\|_2 \frac{2^{\frac{1}{q}-m-\frac{1}{2}}}{\Gamma(m+1)} \left(\sum_{i=0}^{k} \frac{\Gamma^2(i+2m+1)(2m+2i+1)}{(i!)^2} \right)^{\frac{1}{2}} \\
& + \frac{1}{k!} \|f^{(m+k+1)}\|_r \left[\int_{-1}^{1} \left(\int_{-1}^{1} |N_{k+1;p(\cdot)}(x,t)|^{r'} dt \right)^{\frac{q}{r'}} dx \right]^{\frac{1}{q}},
\end{aligned}$$

which implies (23). Theorem 2 is proved. □

Remark 2. It can be proved that for $k = 0$ inequality (23) turns into equality.

5. Some estimates for $A_{m+k+1,m}(p,q,r)$

By applying Hölder's and Minkowski inequalities to the right-hand side of (24) we obtain ($p' = \frac{p}{p-1}$ for $1 < p < \infty$, $p' = 1$ for $p = \infty$, $p' = \infty$ for $p = 1$)

$$\|f^{(m)}\|_q \leqslant \|f\|_p \sum_{i=0}^{k} \frac{\sqrt{\Gamma(i+2m+1)}}{\sqrt{\Gamma(i+1)}} \|X_{i+m}\|_{p'} \cdot \|J_i^{(m,m)}\|_q + \frac{1}{k!}\|f^{(m+k+1)}\|_r \left\{ \int_{-1}^{1} \left[\int_{-1}^{1} |N_{k+1;p(\cdot)}(x,t)|^{r'} dt \right]^{\frac{q}{r'}} dx \right\}^{\frac{1}{q}}. \tag{25}$$

To derive estimates for $A_{m+k+1,m}(p,q,r)$ from (25) we need estimates for $\|X_{i+m}\|_{p'}$ and $\|J_i^{(m,m)}\|_q$.

Lemma 3. *The following estimates for* $\|X_{i+m}\|_{p'}$ are valid.

(1) *If* $p > \frac{4}{3}$, *then*

$$\|X_{i+m}\|_{p'} \leqslant \sqrt{\frac{2i+2m+1}{2}} (i+m)^{-\frac{1}{4}} (i+m+1)^{-\frac{1}{4}} \times \left(\frac{\sqrt{\pi}\,\Gamma\left(-\frac{p'}{4}+1\right)}{\Gamma\left(-\frac{p'}{4}+\frac{3}{2}\right)} \right)^{\frac{1}{p'}}; \tag{26}$$

(2) *if* $p = \frac{4}{3}$, *then*

$$\|X_{i+m}\|_4 \leqslant 2^{-\frac{1}{4}} 3^{\frac{1}{2}} \ln^{\frac{1}{4}} [1 + (i+m)(i+m+1)]; \tag{27}$$

(3) *if* $p < \frac{4}{3}$, *then*

$$\|X_{i+m}\|_{p'} \leqslant 3^{\frac{1}{2}} 2^{\frac{1}{p'}-\frac{1}{2}} \left(\frac{p'}{4} - 1 \right)^{-\frac{1}{p'}} (i+m)^{\frac{1}{2}-\frac{2}{p'}}. \tag{28}$$

Proof. To prove (26) we make use of Martin's inequality for Legendre polynomials (see [7])

$$|P_n(x)| \leqslant \frac{1}{[1 + n(n+1)(1-x^2)]^{\frac{1}{4}}}, \quad n \in N, \ |x| \leqslant 1.$$

The following goes without saying:

$$
\begin{aligned}
||X_{i+m}||_{p'} &= \sqrt{\frac{2i+2m+1}{2}}\left\{\int_{-1}^{1}|P_{i+m}(u)|^{p'}\,du\right\}^{\frac{1}{p'}}\\
&\leqslant \sqrt{2i+2m+1}\times 2^{\frac{1}{p'}-\frac{1}{2}}\left\{\int_0^1 \frac{du}{[1+(i+m)(i+m+1)(1-u^2)]^{\frac{p'}{4}}}\right\}^{\frac{1}{p'}}\\
&= \sqrt{2i+2m+1}\times 2^{-\frac{1}{2}}(i+m)^{-\frac{1}{2p'}}(i+m+1)^{-\frac{1}{2p'}}\\
&\quad\times\left\{\int_1^{1+(i+m)(i+m+1)}[(i+m)(i+m+1)+1-z]^{-\frac{1}{2}}z^{-\frac{p'}{4}}\,dz\right\}^{\frac{1}{p'}}\\
&\leqslant \sqrt{2i+2m+1}\times 2^{-\frac{1}{2}}(i+m)^{-\frac{1}{2p'}}\\
&\quad\times\left\{\int_1^{1+(i+m)(i+m+1)}[1+(i+m)(i+m+1)-z]^{-\frac{1}{2}}(z-1)^{-\frac{p'}{4}}\,dz\right\}^{\frac{1}{p'}}\\
&= \sqrt{2i+2m+1}\times 2^{-\frac{1}{2}}(i+m)^{-\frac{1}{4}}(i+m+1)^{-\frac{1}{4}}\left(\frac{\sqrt{\pi}\Gamma\left(-\frac{p'}{4}+1\right)}{\Gamma\left(-\frac{p'}{4}+\frac{3}{2}\right)}\right)^{\frac{1}{p'}}.
\end{aligned}
$$

To obtain (27) and (28) we start as before but instead of using the estimate $z^{-\frac{p'}{4}} < (z-1)^{-\frac{p'}{4}}$ $(z>1)$ we will apply to the integral

$$\int_1^{1+(i+m)(i+m+1)}[1+(i+m)(i+m+1)-z]^{-\frac{1}{2}}z^{-\frac{p'}{4}}\,dz$$

the following Chebyshev's inequality: if $f, g \in C[a,b]$ and are monotonic in an opposite way, then

$$\int_a^b f(x)\,dx\int_a^b g(x)\,dx \geqslant (b-a)\int_a^b f(x)g(x)\,dx,$$

(see [2, p. 51]). On the supposition of $p<\frac{4}{3}$ we get

$$
\begin{aligned}
&\int_1^{1+(i+m)(i+m+1)}((i+m)(i+m+1)+1-z)^{-\frac{1}{2}}z^{-\frac{p'}{4}}\,dz\\
&\quad\leqslant \frac{2}{\sqrt{(i+m)(i+m+1)}\left(-\frac{p'}{4}+1\right)}\{[1+(i+m)(i+m+1)]^{-\frac{p'}{4}}-1\}\\
&\quad\leqslant \frac{2}{\sqrt{(i+m)(i+m+1)}\left(\frac{p'}{4}-1\right)},
\end{aligned}
$$

which implies

$$\|X_{i+m}\|_{p'} \leqslant \sqrt{\frac{2i+2m+1}{2}}(i+m)^{-\frac{1}{2p'}}(i+m+1)^{-\frac{1}{2p'}} \times \frac{2^{\frac{1}{p'}}}{\left(\frac{p'}{4}-1\right)^{\frac{1}{p'}}}((i+m)(i+m+1))^{-\frac{1}{2p'}}$$

$$\leqslant \sqrt{3}\cdot\frac{2^{\frac{1}{p'}-\frac{1}{2}}}{\left(\frac{p'}{4}-1\right)^{\frac{1}{p'}}}(i+m)^{\frac{1}{2}-\frac{2}{p'}}.$$

Estimate (28) is proved. Estimate (27) can be proved in a similar manner. This completes the proof of Lemma 3. □

Remark 3. All the estimates (26)–(28) give for $\|X_{i+m}\|_{p'}$ the right order of growth with respect to $i+m$ (see [4]).

Lemma 4. *There are following estimates for* $\| J_i^{(m,m)}\|_q$:

(1) *for* $q>2$, $m\in N$ *we have*

$$\| J_i^{(m,m)}\|_q \leqslant \frac{(i+1)^{1-\frac{2}{q}}\pi^{-\frac{3}{q}+\frac{2}{q^2}+1}}{\sqrt{i!}\,2^{-\frac{3}{q}+\frac{2}{q^2}+\frac{1}{2}+m}}\left(\frac{q}{q-2}\right)^{\frac{1}{q}} \times \frac{\sqrt{2m+2i+1}}{\Gamma(m)}\frac{\sqrt{\Gamma(i+2m)}}{\sqrt{i+2m}}; \tag{29}$$

(2) *for* $m=1$, $q=\frac{4}{3}$ *we have*

$$\| J_i^{(1,1)}\|_{\frac{4}{3}} \leqslant \frac{12^{\frac{3}{4}}}{\pi^{\frac{1}{2}}}\ln^{\frac{3}{4}}(i+2)+2^{-\frac{1}{4}}; \tag{30}$$

(3) *for* $m=1$, $q<\frac{4}{3}$ *we have*

$$\| J_i^{(1,1)}\|_q \leqslant \frac{2^{\frac{1}{q}}3^{\frac{3}{4}}}{\pi^{\frac{1}{2}}\left(1-\frac{3q}{4}\right)}; \tag{31}$$

(4) *for* $m=1$, $q>\frac{4}{3}$ *we have*

$$\| J_i^{(1,1)}\|_q \leqslant \left(\frac{2^{\frac{1}{q}}3^{\frac{3}{4}}}{\pi^{\frac{1}{2}}\left(\frac{3q}{4}-1\right)^{\frac{1}{q}}}+2^{\frac{1}{q}-1}\right)(i+2)^{\frac{3}{2}-\frac{2}{q}}; \tag{32}$$

(5) *for* $m \geqslant 2$, $1 \leqslant q \leqslant 2$ *we have*

$$\| J_i^{(m,m)} \|_q \leqslant 2^{1-\frac{1}{q}+\frac{-4m^2+16m+25}{2(2m+1)}} (m!)^{\frac{3-2m}{2m+1}} \frac{2m+1}{2m-3} \pi^{-\frac{2}{2m+1}} \times \frac{\sqrt{2i+2m+1}\sqrt{\Gamma(i+2m+1)}}{\sqrt{\Gamma(i+1)}} \left[\frac{\Gamma\left(i+m+\frac{1}{2}\right)}{\Gamma(i+2m+1)} \right]^{\frac{4}{2m+1}}. \tag{33}$$

Proof. (1) To prove (29) we make use of the following inequality which was proved in [1]:

if $f \in H_n$, $n \in N$, $1 \leqslant q \leqslant \infty$ then

$$\|f'\|_q \leqslant \|T_n'\|_q \cdot \|f\|_C, \quad T_n(x) = \cos(n \arccos x). \tag{34}$$

In case of $2 < q < \infty$ it follows from (34) that for $f \in H_n$

$$\|f'\|_q \leqslant \frac{\pi^{-\frac{3}{q}+\frac{2}{q^2}+1}}{2^{-\frac{3}{q}+\frac{2}{q^2}+1}} \left(\frac{q}{q-2} \right)^{\frac{1}{q}} n^{2-\frac{2}{q}} \|f\|_C. \tag{35}$$

Taking (35) into consideration we obtain

$$\begin{aligned}
\| J_i^{(m,m)} \|_q &= \frac{\sqrt{i!}}{\sqrt{\Gamma(i+2m+1)}} \|(X_{i+m}^{(m-1)})'\|_q \\
&\leqslant \frac{\sqrt{i!}}{\sqrt{\Gamma(i+2m+1)}} \frac{\pi^{-\frac{3}{q}+\frac{q}{q^2}+1}}{2^{-\frac{3}{q}+\frac{2}{q^2}+1}} \left(\frac{q}{q-2} \right)^{\frac{1}{q}} \times (i+1)^{2-\frac{2}{q}} \|X_{i+m}^{(m-1)}\|_C \\
&= \frac{\sqrt{i!}}{\sqrt{\Gamma(i+2m+1)}} \frac{\pi^{-\frac{3}{q}+\frac{2}{q^2}+1}}{2^{-\frac{3}{q}+\frac{2}{q^2}+1}} \left(\frac{q}{q-2} \right)^{\frac{1}{q}} (i+1)^{2-\frac{2}{q}} X_{i+m}^{(m-1)}(1) \\
&= \frac{(i+1)^{1-\frac{2}{q}}}{\sqrt{i!}} \frac{\pi^{-\frac{3}{q}+\frac{2}{q^2}+1}}{2^{-\frac{3}{q}+\frac{2}{q^2}+1+m}} \left(\frac{q}{q-2} \right)^{\frac{1}{q}} \frac{\sqrt{2m+2i+1}}{\Gamma(m)} \frac{\sqrt{\Gamma(i+2m)}}{\sqrt{i+2m}}.
\end{aligned}$$

Estimate (29) is proved. □

(2) In case of $m = 1$, $q = \frac{4}{3}$ we take into account that

$$J_i^{(1,1)}(x) = \frac{\sqrt{2i+3}}{\sqrt{2}\sqrt{i+1}\sqrt{i+2}} P_{i+1}'(x) \tag{36}$$

and

$$(1-x^2)^{\frac{3}{4}} |P_n'(x)| < \frac{3^{\frac{3}{4}}}{\pi^{\frac{1}{2}}} \sqrt{n}, \quad x \in [-1, 1] \tag{37}$$

(see [8, p. 68]). By setting $\varepsilon = (i+2)^{-2}$, applying (36), (37) and taking into consideration that $||P'_{i+1}||_{C[-1,1]} = P'_{i+1}(1) = \frac{(i+1)(i+2)}{2}$, we get

$$\begin{aligned}
|| J_i^{(1,1)}||_{\frac{4}{3}} &= \frac{\sqrt{2i+3}}{\sqrt{2}\sqrt{i+1}\sqrt{i+2}}\left\{\int_{-1}^{1} |P'_{i+1}(x)|^{\frac{4}{3}}\,dx\right\}^{\frac{3}{4}} \\
&< \frac{1}{\sqrt{i+1}}\left\{2\int_0^1 |P'_{i+1}(x)|^{\frac{4}{3}}\,dx\right\}^{\frac{3}{4}} \\
&= \frac{2^{\frac{3}{4}}}{\sqrt{i+1}}\left\{\int_0^{1-\varepsilon} |P'_{i+1}(x)|^{\frac{4}{3}}\,dx + \int_{1-\varepsilon}^{1} |P'_{i+1}(x)|^{\frac{4}{3}}\,dx\right\}^{\frac{3}{4}} \\
&\leqslant \frac{2^{\frac{3}{4}}}{\sqrt{i+1}}\left\{\left[\int_0^{1-\varepsilon} |P'_{i+1}(x)|^{\frac{4}{3}}\,dx\right]^{\frac{3}{4}} + \left[\int_{1-\varepsilon}^{1} |P'_{i+1}(x)|^{\frac{4}{3}}\,dx\right]^{\frac{3}{4}}\right\} \\
&\leqslant \frac{2^{\frac{3}{4}}}{\sqrt{i+1}}\left\{\frac{3^{\frac{3}{4}}\sqrt{i+1}}{\pi^{\frac{1}{2}}}\left[\int_0^{1-\varepsilon} (1-x^2)^{-1}\,dx\right]^{\frac{3}{4}} + \varepsilon^{\frac{3}{4}}\frac{(i+1)(i+2)}{2}\right\} \\
&\leqslant \frac{6^{\frac{3}{4}}}{\pi^{\frac{1}{2}}}\left[\int_0^{1-\varepsilon} (1-x)^{-1}\,dx\right]^{\frac{3}{4}} + 2^{-\frac{1}{4}}\sqrt{i+1}(i+2)\varepsilon^{\frac{3}{4}} \\
&= \frac{6^{\frac{3}{4}}}{\pi^{\frac{1}{2}}}(-\ln\varepsilon)^{\frac{3}{4}} + 2^{-\frac{1}{4}}\sqrt{i+1}(i+2)\varepsilon^{\frac{3}{4}} \leqslant \frac{12^{\frac{3}{4}}}{\pi^{\frac{1}{2}}}\ln^{\frac{3}{4}}(i+2) + 2^{-\frac{1}{4}}.
\end{aligned}$$

The inequality (30) is proved.

(3) To prove (31) it suffices to make use of (36) and (37) when estimating $|| J_i^{(1,1)}||_q$.

(4) Proof of (32) is similar to the proof of (30).

(5) To prove (33) we consider the case $q = 1$ first. There is the following relationship between the polynomial $J_i^{(m,m)}(x)$ and the Legendre function $P_{i+m}^m(x)$:

$$J_i^{(m,m)}(x) = \sqrt{\frac{i!}{\Gamma(i+2m+1)}}\frac{\sqrt{2i+2m+1}}{\sqrt{2}}(1-x^2)^{-\frac{m}{2}}P_{i+m}^m(x). \tag{38}$$

We will use the following estimates for the Legendre function $P_n^m(x)$:

$$|(1-x^2)^{\frac{1}{4}}P_n^m(x)| < \frac{2^{m+1}\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n-m+1)}, \quad m,n \in N, \quad x \in [-1,1], \tag{39}$$

$$|P_n^m(x)| \leqslant (1-x^2)^{\frac{m}{2}}\frac{2^{-m}(n+m)!}{m!(n-m)!}, \quad 0 \leqslant m \leqslant n, \ m,n \in W, \ x \in [-1,1]; \tag{40}$$

(see [6]). Making use of (39) and (40) we obtain $(0<\varepsilon<1)$

$$\begin{aligned}
&\int_{-1}^{1} |J_i^{(m,m)}(x)|\,dx \\
&= \sqrt{\frac{i!}{\Gamma(i+2m+1)}}\sqrt{\frac{2i+2m+1}{2}}\int_{-1}^{1}(1-x^2)^{-\frac{m}{2}}|P_{i+m}^m(x)|\,dx \\
&= \frac{\sqrt{i!}\,\sqrt{2i+2m+1}}{\sqrt{\Gamma(i+2m+1)}\,\sqrt{2}}\bigg(\int_{-1+\varepsilon}^{1-\varepsilon}(1-x^2)^{-\frac{m}{2}-\frac{1}{4}}(1-x^2)^{\frac{1}{4}}|P_{i+m}^m(x)|\,dx \\
&\quad + 2\int_{1-\varepsilon}^{1}(1-x^2)^{-\frac{m}{2}}|P_{i+m}^m(x)|\,dx\bigg) \\
&\leqslant \frac{\sqrt{i!}\,\sqrt{2i+2m+1}}{\sqrt{\Gamma(i+2m+1)}\sqrt{2}}\bigg(\frac{2^{m+1}\Gamma\left(i+m+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(i+1)} \\
&\quad \times 2\int_{0}^{1-\varepsilon}(1-x^2)^{-\frac{m}{2}-\frac{1}{4}}\,dx + \frac{2^{-m+1}(i+2m)!}{m!i!}\varepsilon\bigg) \\
&\leqslant \frac{\sqrt{i!}\sqrt{2i+2m+1}}{\sqrt{\Gamma(i+2m+1)}\sqrt{2}}\bigg(\frac{2^{m+2}\Gamma\left(i+m+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(i+1)}\int_{0}^{1-\varepsilon}(1-x)^{-\frac{m}{2}-\frac{1}{4}}\,dx \\
&\quad + \frac{2^{-m+1}(i+2m)!}{m!i!}\varepsilon\bigg) \\
&= \frac{\sqrt{i}\sqrt{2i+2m+1}}{\sqrt{\Gamma(i+2m+1)}\sqrt{2}}\left(\frac{2^{m+2}\Gamma\left(i+m+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(i+1)}\left(-\frac{\varepsilon^{-\frac{m}{2}+\frac{3}{4}}}{-\frac{m}{2}+\frac{3}{4}}+\frac{1}{-\frac{m}{2}+\frac{3}{4}}\right)\right. \\
&\quad \left. + \frac{2^{-m+1}(i+2m)!}{m!i!}\varepsilon\right) \\
&< \frac{\sqrt{i}\sqrt{2i+2m+1}}{\sqrt{\Gamma(i+2m+1)}\sqrt{2}}\left(\frac{2^{m+2}\Gamma\left(i+m+\frac{1}{2}\right)\varepsilon^{-\frac{m}{2}+\frac{3}{4}}}{\sqrt{\pi}\Gamma(i+1)\left(\frac{m}{2}-\frac{3}{4}\right)}+\frac{2^{-m+1}(i+2m)!}{m!i!}\varepsilon\right) \\
&= \frac{\sqrt{i!}\sqrt{2i+2m+1}}{\sqrt{\Gamma(i+2m+1)}\sqrt{2}}\,\varphi(\varepsilon).
\end{aligned} \tag{41}$$

By minimizing $\varphi(\varepsilon)$ over $0<\varepsilon<1$ we derive from (41) that

$$\begin{aligned}
\int_{-1}^{1} |J_i^{(m,m)}(x)|\,dx \leqslant{}& 2^{\frac{-4m^2+16m+25}{2(2m+1)}}(m!)^{\frac{3-2m}{2(2m+1)}}\frac{2m+1}{2m-3}\pi^{-\frac{2}{2m+1}} \\
&\times \frac{\sqrt{2i+2m+1}\sqrt{\Gamma(i+2m+1)}}{\sqrt{\Gamma(i+1)}}\left[\frac{\Gamma\left(i+m+\frac{1}{2}\right)}{\Gamma(i+2m+1)}\right]^{\frac{4}{2m+1}},
\end{aligned} \tag{42}$$

so that for $q=1$ estimate (33) is proved. To complete proof of (33) we can use (42) and the following result: $\forall f\in H_n$, $1\leqslant q\leqslant\infty$, we have

$$||f||_q\leqslant 2^{1-\frac{1}{q}}n^{2\left(1-\frac{1}{q}\right)}||f||_1. \tag{43}$$

Estimate (33) is an immediate consequence of (42) and (43). This completes the proof of Lemma 4. □

Remark 4. All the estimates (29)–(33) give for $||J_i^{(m,m)}||_q$ the right order of growth with respect to i (see [4]).

When deriving estimates for $A_{m+k+1,m}(p,q,r)$ from (25) and Lemmas 3 and 4, we have to differentiate between 15 cases. We will list estimates for $A_{m+k+1,m}(p,q,r)$ such that right-hand side gives for $k=0$ the right order of growth with respect to m (see (2)). Besides, we list some estimates for $A_{k+2,1}(p,q,r)$.

Theorem 3. (1) *For $p>\frac{4}{3}$, $q>2$ the following estimate is valid ($m\in N$, $k\in W$):*

$$A_{m+k+1,m}(p,q,r)\leqslant \Gamma^{\frac{1}{p'}}\left(-\frac{p'}{4}+1\right)\Gamma^{-\frac{1}{p'}}\left(-\frac{p'}{4}+\frac{3}{2}\right)2^{\frac{3}{q}-\frac{2}{q^2}-1}\pi^{\frac{1}{2p'}-\frac{3}{q}+\frac{2}{q^2}+1}$$
$$\times\left(\frac{q}{q-2}\right)^{\frac{1}{q}}\frac{1}{2^m\Gamma(m)}\sum_{i=0}^{k}(i!)^{-1}\Gamma(i+2m)$$
$$\times(2i+2m+1)(i+1)^{1-\frac{2}{q}}(i+m)^{-\frac{1}{4}}(i+m+1)^{-\frac{1}{4}}.$$

(2) *For $p>\frac{4}{3}$, $1\leqslant q\leqslant 2$, $m\geqslant 2$, $m\in N$, $k\in W$ we have*

$$A_{m+k+1,m}(p,q,r)\leqslant \Gamma^{\frac{1}{p'}}\left(-\frac{p'}{4}+1\right)\Gamma^{-\frac{1}{p'}}\left(-\frac{p'}{4}+\frac{3}{2}\right)\pi^{\frac{1}{2p'}-\frac{1}{2m+1}}$$
$$\times 2^{\frac{1}{2}-\frac{1}{q}+\frac{-4m^2+16m+25}{2(2m+1)}}\sum_{i=0}^{k}\frac{\Gamma(i+2m+1)(2i+2m+1)}{i!}$$
$$\times(i+m)^{-\frac{1}{4}}(i+m+1)^{-\frac{1}{4}}$$
$$\times\left[\frac{\Gamma\left(i+m+\frac{1}{2}\right)}{\Gamma(i+2m+1)}\right]^{\frac{4}{2m+1}}\times(m!)^{\frac{3-2m}{2m+1}}\frac{2m+1}{2m-3}.$$

(3) *For $p>\frac{4}{3}$, $q<\frac{4}{3}$, $k\in W$ we have*

$$A_{k+2,1}(p,q,r)\leqslant\frac{\Gamma^{\frac{1}{p'}}\left(-\frac{p'}{4}+1\right)\Gamma^{-\frac{1}{p'}}\left(-\frac{p'}{4}+\frac{3}{2}\right)\pi^{-\frac{1}{2p}}2^{\frac{1}{q}-1}\,3^{\frac{3}{4}}}{1-\frac{3q}{4}}(k+1)\,(k+4).$$

(4) *For* $p<\frac{4}{3}$, $\frac{4}{3}<q\leqslant 2$, $k\in W$ *we have*

$$A_{k+2,1}(p,q,r)\leqslant 3^{\frac{1}{2}}2^{\frac{1}{q}+\frac{1}{p'}-\frac{1}{2}}\left(\frac{p'}{4}-1\right)^{-\frac{1}{p'}} \times\left(\frac{3^{\frac{3}{4}}}{\pi^{\frac{1}{2}}\left(\frac{3q}{4}-1\right)^{\frac{1}{q}}}+2^{-1}\right)\frac{(k+3)^{\frac{2}{p}+2-\frac{2}{q}}}{\frac{2}{p}+2-\frac{2}{q}}.$$

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